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INTEGRAL INEQUALITIES FOR EXTENDED HARMONIC CONVEX FUNCTIONS

Muhammad Aslam Noor[1*], Khalida Inayat Noor[1], Sabah Iftikhar[1]

[1]Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

Abstract. In this paper, we introduce a new class of extended harmonic convex functions. Some new Hermite-Hadamard type inequalities are derived. Some special cases are discussed. Results represent significant refinement and improvement of the previous results. Ideas of this paper may stimulate further research.

Keywords: *harmonic convex functions, harmonic h-convex functions, harmonic (s,m) -convex functions, Hermite-Hadamard type inequality.*

AMS Subject Classification: *26D15, 26D10, 90C23.*

****Corresponding Author:*** *Prof. Muhammad Aslam Noor, Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan, e-mail:noormaslam@gmail.com*

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1. Introduction

Convexity theory has become a rich source of inspiration in pure and applied sciences. This theory had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provide us a unified and general framework for studying a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions and their variant forms, see [1-27] and the references therein.

In recent years, the convex sets and convex functions have been extended and generalized in several directions using novel and innovative ideas and techniques. Varosanec [22] introduced the class of *h*-convex functions. This class of functions unifies various classes of convex functions and is being used to discuss several concepts in a unified manners. Toader [25] defined the m-convexity, an intermediate between the usual convexity and starshaped property. Park [23] considered the class of (s,m)-convex functions. An important class of convex functions, which is called harmonic convex function, was introduced and studied by Anderson et al. [1] and Iscan [12].

We would like to emphasize that (s,m)-convex functions and harmonic functions are two distinct classes of convex functions. It is natural to introduce a new class of convex functions, which unifies these concepts. Inspired and motivated by the ongoing research activities in this dynamic field, we introduce a new class of convex functions, which is called extended harmonic (h,s,m)-

convex function. One can easily show that extended harmonic (h,s,m)-convex functions include Godunova-Levin harmonic convex functions, extended harmonic s-convex functions and harmonic m-convex functions as special cases. We also obtain several new Hermite-Hadamard type inequalities. Our results include several previously known and new results as special cases. It is expected that results obtained in this paper may inspire the readers to discover new applications of the extended harmonic (h,s,m)-convex functions in various branches of pure and applied sciences. This is another direction of future research.

2. Preliminaries

First of all, we recall the following basic concepts. To convey an idea of the harmonic convex set, we include the formal defitiniton of the harmonic convex set.

Definition 1. [12] A set $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ is said to be a harmonic convex set, if

$$\frac{xy}{tx+(1-t)y}\in I,\qquad \forall x,y\in I, t\in[0,1].$$

We now consider the new concept of extended harmonic $(h,s,m)-$convex functions, which is the main motivation of this paper.

Definition 2. Let $h:J=[0,1]\to\mathbb{R}$ a nonnegative function. A function $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be extended harmonic (h,s,m)-convex function in second sense, where $s\in[-1,1]$, $m\in(0,1]$ and $a<b, ma\in I$, if

$$f\left(\frac{mxy}{tx+m(1-t)y}\right)\le h((1-t)^s)f(x)+mh(t^s)f(y),\quad \forall x,y\in I, t\in(0,1).$$

For $t=\frac{1}{2}$, we have

$$f\left(\frac{2mxy}{x+my}\right)\le h\left(\frac{1}{2^s}\right)\left[f(x)+mf(y)\right],\quad \forall x,y\in I,$$

The function f is called Jensen type extended harmonic (h,s,m)-convex function.

Now we discuss some special cases of extended harmonic (h,s,m) convex function.

I. If $h=t^s$, $s=-1$ and $m=1$ in Definition 2, then it reduces to the Definition of Godunova-Levin harmonic convex functions.

Definition 3.[16] A function $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be Godunova-Levin harmonic convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\le\frac{1}{1-t}f(x)+\frac{1}{t}f(y),\qquad \forall x,y\in I, t\in(0,1).$$

II. If $h=t^s$ in Definition 2, then it reduces to the Definition of harmonic (s,m)-convex functions.

Definition 4. [2] A function $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonic (s,m)-convex function in second sense, where $s\in(0,1]$ and $m\in(0,1]$, if

$$f\left(\frac{mxy}{tx+m(1-t)y}\right)\le (1-t)^s f(x)+mt^s f(y), \qquad \forall x,y\in I, t\in[0,1].$$

We remark that if $t=1$, $y=a$, then $f(my)\le mf(a)$. In this case, we say that the function f is subhomogenous.

III. If $m=1$ in Definition 2, then it reduces to the Definition of extended harmonic (h,s)-convex functions.

Definition 5. A function $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be extended harmonic (h,s)-convex function in second sense, where $s\in[-1,1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\le h((1-t)^s) f(x)+h(t^s) f(y), \qquad \forall x,y\in I, t\in(0,1).$$

IV. If $s=1$ in Definition 2, then it reduces to the Definition of harmonic (h,m)-convex functions.

Definition 6. A function $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonic (h,m)-convex function, where $m\in(0,1]$, if

$$f\left(\frac{mxy}{tx+m(1-t)y}\right)\le h(1-t) f(x)+mh(t) f(y), \qquad \forall x,y\in I, t\in[0,1].$$

V. If $s=1$ and $m=1$ in Definition 2, then it reduces to the Definition of harmonic h-convex functions.

Definition 7. [15]. A function $f: I=[a,b]\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be a harmonic h-convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\le h(1-t) f(x)+h(t) f(y), \qquad \forall x,y\in I, t\in[0,1].$$

Thus it is clear that the class of extended harmonic (h,s,m) convex functions is quite general and include several new classes of convex functions as special cases.

Definition 8. [22] Two functions f,g are said to be similarly ordered (f is g-monotone), if and only if,

$$\langle f(x)-f(y), g(x)-g(y)\rangle \ge 0, \qquad \forall x,y\in\mathbb{R}^n.$$

Now we show that the product of two extended harmonic (h,s,m)-convex functions is again extended harmonic (h,s,m)-convex function, which is the main motivation of our next result.

Lemma 1. If f and g are two similarly ordered extended harmonic (h,s,m)-convex functions, where $mh(t^s)+h((1-t)^s)\le 1$, then the product fg is again a extended harmonic (h,s,m)-convex function.

Proof. Let f and g be two similarly ordered extended harmonic (h,s,m)-convex functions. Then

$$f\left(\frac{mxy}{tx+m(1-t)y}\right)g\left(\frac{mxy}{tx+m(1-t)y}\right)$$

$$\leq [h((1-t)^s) f(x) + mh(t^s) f(y)][h((1-t)^s) g(x) + mh(t^s) g(y))]$$

$$= [h((1-t)^s)]^2 f(x)g(x) + mh(t^s)h((1-t)^s)[f(x)g(y) + f(y)g(x)]$$
$$+m^2[h(t^s)]^2 f(y)g(y)$$
$$\leq [h((1-t))^s]^2 f(x)g(x) + mh(t^s)h((1-t)^s)[f(x)g(x) + f(y)g(y)]$$
$$+m^2[h(t^s)]^2 f(y)g(y)$$
$$= [h((1-t)^s) f(x)g(x) + mh(t^s) f(y)g(y)][mh(t^s) + h((1-t)^s)]$$
$$\leq h((1-t)^s) f(x)g(x) + mh(t^s) f(y)g(y), (1)$$

where we used the fact that $mh(t^s) + h((1-t)^s) \leq 1.$
This shows that product of two similarly ordered extended harmonic (h,s,m)-convex functions is again a extended harmonic (h,s,m)-convex function.

We also need the following known fact, which plays a crucial role in the derivation of main results.

Remark 1. Let $I = [a,b] \subset \mathbb{R} \setminus \{0\}$ and consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is extended harmonic (h,s,m)-convex on $[a,b]$, if and only if, g is extended (h,s,m)-convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

3. Main Results

In this section, we obtain Hermite-Hadamard inequalities for harmonic (h,s,m)-convex function.

Theorem 1. Let $f : I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic (h,s,m)- convex function, where $s \in (-1,1],\ m \in (0,1]$ and $a,b \in I$ with $a < b,\ ma, \frac{b}{m} \in I.$ If $f \in L[ma, \frac{b}{m}]$, then $f : I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$$\frac{1}{h\left(\frac{1}{2^s}\right)} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{x^2} \mathrm{d}x$$

$$\leq \left[f(a) + mf\left(\frac{b}{m}\right) + m^2 f\left(\frac{b}{m^2}\right) + mf\left(\frac{a}{m}\right)\right] \int_0^1 h(t^s) \mathrm{d}t. (2)$$

Proof. Let f be harmonic (h,s,m)-convex function with $t = \frac{1}{2}$. Then

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2^s}\right)\left[f(x) + mf\left(\frac{y}{m}\right)\right].$$

Taking $x=\dfrac{ab}{ta+(1-t)b}$ and $y=\dfrac{ab}{(1-t)a+tb}$, we have

$$f\left(\frac{2ab}{a+b}\right)\le h\left(\frac{1}{2^s}\right)\left[f\left(\frac{ab}{ta+(1-t)b}\right)+mf\left(\frac{ab}{m(1-t)a+mtb}\right)\right]$$

$$=h\left(\frac{1}{2^s}\right)\left[\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+m\int_0^1 f\left(\frac{ab}{m(1-t)a+mtb}\right)\mathrm{d}t\right]$$

$$\le h\left(\frac{1}{2^s}\right)\left[\begin{array}{l}h((1-t)^s)f(a)+mh(t^s)f\left(\dfrac{b}{m}\right)+m^2h((1-t)^s)f\left(\dfrac{b}{m^2}\right)\\ +mh(t^s)f\left(\dfrac{a}{m}\right)\end{array}\right]$$

$$=h\left(\frac{1}{2^s}\right)\left[f(a)+mf\left(\frac{b}{m}\right)+m^2f\left(\frac{b}{m^2}\right)+mf\left(\frac{a}{m}\right)\right]\int_0^1 h(t^s)\mathrm{d}t.$$

Taking into account that

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t=\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x$$

and

$$\int_0^1 f\left(\frac{ab}{m(1-t)a+mtb}\right)\mathrm{d}t=\frac{ab}{m(b-a)}\int_{\frac{a}{m}}^{\frac{b}{m}}\frac{f(\tau)}{\tau^2}\mathrm{d}\tau=\frac{ab}{b-a}\int_a^b\frac{f\left(\frac{x}{m}\right)}{x^2}\mathrm{d}x.$$

This implies

$$f\left(\frac{2ab}{a+b}\right)\le h\left(\frac{1}{2^s}\right)\frac{ab}{(b-a)}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{x^2}\mathrm{d}x$$

$$\le h\left(\frac{1}{2^s}\right)\left[f(a)+mf\left(\frac{b}{m}\right)+m^2f\left(\frac{b}{m^2}\right)+mf\left(\frac{a}{m}\right)\right]\int_0^1 h(t^s)\mathrm{d}t.$$

Corollary 1. Under the assumptions of Theorem 1 and $h(t^s)=t^s$, we have

$$f\left(\frac{2ab}{a+b}\right)\le\frac{ab}{2^s(b-a)}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{x^2}\mathrm{d}x$$

$$\le\frac{\left[f(a)+mf\left(\frac{b}{m}\right)+m^2f\left(\frac{b}{m^2}\right)+mf\left(\frac{a}{m}\right)\right]}{2^s(s+1)}.$$

Theorem 2. Let $f:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m)- convex function, where $s\in(-1,1]$, $m\in(0,1]$ and $a,b\in I$ with $a<b,\ ma\in I$. If $f\in L[ma,b]$, then

$$\frac{2mab}{m+1}\left[\int_a^{mb}\frac{f(x)}{x^2}\mathrm{d}x+\frac{mb-a}{b-ma}\int_{ma}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right]\le (mb-a)[f(a)+f(b)]$$
$$\times\int_0^1[h((1-t)^s)+h(t^s)]\mathrm{d}t.$$

Proof. Let f be harmonic (h,s,m)-convex function. Then we have

$$f\left(\frac{mab}{ta+m(1-t)b}\right)\le h((1-t)^s)f(a)+mh(t^s)f(b),$$
$$f\left(\frac{mab}{(1-t)a+mtb}\right)\le h(t^s)f(a)+mh((1-t)^s)f(b),$$
$$f\left(\frac{mab}{mta+(1-t)b}\right)\le mh((1-t)^s)f(a)+h(t^s)f(b)$$

and

$$f\left(\frac{mab}{m(1-t)a+tb}\right)\le mh(t^s)f(a)+h((1-t)^s)f(b).$$

Adding the above inequalities, we have

$$f\left(\frac{mab}{ta+m(1-t)b}\right)+f\left(\frac{mab}{(1-t)a+mtb}\right)+f\left(\frac{mab}{mta+(1-t)b}\right),$$
$$f\left(\frac{mab}{m(1-t)a+tb}\right)\le (m+1)[f(a)+f(b)][h((1-t)^s)+h(t^s)].$$

Integrating over $t\in[0,1]$, we obtain

$$\int_0^1 f\left(\frac{mab}{ta+m(1-t)b}\right)\mathrm{d}t+\int_0^1 f\left(\frac{mab}{(1-t)a+mtb}\right)\mathrm{d}t$$
$$+\int_0^1 f\left(\frac{mab}{mta+(1-t)b}\right)\mathrm{d}t+\int_0^1 f\left(\frac{mab}{m(1-t)a+tb}\right)\mathrm{d}t$$
$$\le (m+1)[f(a)+f(b)]\int_0^1[h((1-t)^s)+h(t^s)]\mathrm{d}t.$$

This implies

$$\frac{2mab}{m+1}\left[\int_a^{mb}\frac{f(x)}{x^2}\mathrm{d}x+\frac{mb-a}{b-ma}\int_{ma}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right]\le (mb-a)[f(a)+f(b)]$$
$$\times\int_0^1[h((1-t)^s)+h(t^s)]\mathrm{d}t,$$

which is the required result.

Corollary 2. Under the assumptions of Theorem 2 and $h(t^s)=t^s$, we have

$$\frac{mab}{m+1}\left[\int_a^{mb}\frac{f(x)}{x^2}\mathrm{d}x+\frac{mb-a}{b-ma}\int_{ma}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right]\le (mb-a)\frac{f(a)+f(b)}{s+1}.$$

Theorem 3. Let $f,g:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m_1)-convex function and harmonic (h,s,m_2)-convex function, respectively, where $s\in(-1,1]$ and $m\in(0,1]$ and with $a,\frac{b}{m}\in I$. If $f\in L[a,\frac{b}{m}]$, then

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \le \min\{M_1(a,b), M_2(a,b)\},$$

where

$$M_1(a,b) = \left[f(a)g(a) + m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\int_0^1 [h(t^s)]^2\mathrm{d}t$$
$$+\left[m_1 f\left(\frac{b}{m_1}\right)g(a) + m_2 f(a)g\left(\frac{b}{m_2}\right)\right]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t, \tag{3}$$

$$M_2(a,b) = \left[f(b)g(b) + m_1m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 [h(t^s)]^2\mathrm{d}t$$
$$+\left[m_1 f\left(\frac{a}{m_1}\right)g(b) + m_2 f(b)g\left(\frac{a}{m_2}\right)\right]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t.$$

Proof. Let f, g be harmonic (h,s,m_1)-convex function and harmonic (h,s,m_2)-convex function, respectively. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \le h((1-t)^s)f(a) + m_1h(t^s)f\left(\frac{b}{m_1}\right),$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \le h((1-t)^s)g(a) + m_2h(t^s)g\left(\frac{b}{m_2}\right).$$

Now consider

$$f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)$$
$$\le \left[h((1-t)^s)f(a) + m_1h(t^s)f\left(\frac{b}{m_1}\right)\right]\left[h((1-t)^s)g(a) + m_2h(t^s)g\left(\frac{b}{m_2}\right)\right]$$
$$= [h((1-t)^s)]^2[f(a)g(a)] + m_1h(t^s)h((1-t)^s)f\left(\frac{b}{m_1}\right)g(a)$$
$$+ m_2h(t^s)h((1-t)^s)f(a)g\left(\frac{b}{m_2}\right) + m_1m_2[h(t^s)]^2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right].$$

Integrating over $[0,1]$, we have

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t$$
$$\le [f(a)g(a)]\int_0^1 [h((1-t)^s)]^2\mathrm{d}t + m_2 f(a)g\left(\frac{b}{m_2}\right)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t$$
$$+ m_1 f\left(\frac{b}{m_1}\right)g(a)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t + m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\int_0^1 [h(t^s)]^2\mathrm{d}t$$

$$= \left[f(a)g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \right] \int_0^1 [h(t^s)]^2 \mathrm{d}t$$
$$+ \left[m_1 f\left(\frac{b}{m_1}\right) g(a) + m_2 f(a) g\left(\frac{b}{m_2}\right) \right] \int_0^1 h(t^s) h((1-t)^s) \mathrm{d}t.$$

This implies

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \le \left[f(a)g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \right] \int_0^1 [h(t^s)]^2 \mathrm{d}t$$
$$+ \left[m_1 f\left(\frac{b}{m_1}\right) g(a) + m_2 f(a) g\left(\frac{b}{m_2}\right) \right] \int_0^1 h(t^s) h((1-t)^s) \mathrm{d}t.$$

Similarly, we obtain

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \le \left[f(b)g(b) + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) \right] \int_0^1 [h(t^s)]^2 \mathrm{d}t$$
$$+ \left[m_1 f\left(\frac{a}{m_1}\right) g(b) + m_2 f(b) g\left(\frac{a}{m_2}\right) \right] \int_0^1 h(t^s) h((1-t)^s) \mathrm{d}t.$$

Hence

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \le \min\{M_1(a,b), M_2(a,b)\},$$

which is the required result.

Corollary 3. Under the assumptions of Theorem 3 and $h(t^s) = t^s$, we have

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x \le \min\{M_1(a,b), M_2(a,b)\},$$

where

$$M_1(a,b) = \frac{f(a)g(a) + m_1 m_2 \left[f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \right]}{2s+1}$$
$$+ \beta(s+1, s+1) \left[m_1 f\left(\frac{b}{m_1}\right) g(a) + m_2 f(a) g\left(\frac{b}{m_2}\right) \right],$$

$$M_2(a,b) = \frac{f(b)g(b) + m_1 m_2 \left[f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) \right]}{2s+1}$$
$$+ \beta(s+1, s+1) \left[m_1 f\left(\frac{a}{m_1}\right) g(b) + m_2 f(b) g\left(\frac{a}{m_2}\right) \right].$$

Theorem 4. Let $f: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m)-convex function, where $s\in(-1,1]$, $m\in(0,1]$, $a,b\in I$ with $a<b$, $ma,\frac{b}{m}\in I$. If $f\in L[a,b]$, then

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\le\min\left\{\left[f(a)+mf\left(\frac{b}{m}\right)\right]\int_0^1 h(t^s)\mathrm{d}t,\left[mf\left(\frac{a}{m}\right)+f(b)\right]\int_0^1 h(t^s)\mathrm{d}t\right\}.$$

Proof. Let f be harmonic (h,s,m)-convex function. Then

$$f\left(\frac{mab}{ta+m(1-t)b}\right)\le h((1-t)^s)f(a)+mh(t^s)f(b),\quad\forall a,b\in I,t\in(0,1).$$

which gives

$$f\left(\frac{ab}{ta+(1-t)b}\right)\le h((1-t)^s)f(a)+mh(t^s)f\left(\frac{b}{m}\right)$$

and

$$f\left(\frac{ab}{tb+(1-t)a}\right)\le mh(t^s)f\left(\frac{a}{m}\right)+h((1-t)^s)f(b).$$

Integrating on $[0,1]$ we obtain

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\le\left[f(a)+mf\left(\frac{b}{m}\right)\right]\int_0^1 h(t^s)\mathrm{d}t,$$

and

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\le\left[mf\left(\frac{a}{m}\right)+f(b)\right]\int_0^1 h(t^s)\mathrm{d}t.$$

This implies

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\le\min\left\{\left[f(a)+mf\left(\frac{b}{m}\right)\right]\int_0^1 h(t^s)\mathrm{d}t,\left[mf\left(\frac{a}{m}\right)+f(b)\right]\int_0^1 h(t^s)\mathrm{d}t\right\}.$$

Corollary 4. Under the assumptions of Theorem 4 and $h(t^s)=t^s$, we have

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\le\min\left\{\frac{mf\left(\frac{a}{m}\right)+f(b)}{s+1},\frac{f(a)+mf\left(\frac{b}{m}\right)}{s+1}\right\}.$$

Theorem 5. Let $f,g: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m_1)-convex function and harmonic (h,s,m_2)-convex function, respectively, where $s\in(-1,1]$, $m\in(0,1]$, $a,b\in I$ with $a<b$, $ma,\frac{b}{m}\in I$. If $fg\in L[ma,\frac{b}{m}]$, then

$$\left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}}h\left(\left(x-\frac{1}{b}\right)^s\right)\left[f(a)g\left(\frac{1}{x}\right)+g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x+$$

$$\left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}}h\left(\left(\frac{1}{a}-x\right)^s\right)\left[m_1f\left(\frac{b}{m_1}\right)g\left(\frac{1}{x}\right)+m_2g\left(\frac{b}{m_2}\right)f\left(\frac{1}{x}\right)\right]\mathrm{d}x$$

$$\leq M_1(a,b)+\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x,$$

where $M_1(a,b)$ is given by (3).

Proof. Let f,g be harmonic (h,s,m_1)-convex function and harmonic (h,s,m_2)-convex function, respectively. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right)\leq h((1-t)^s)f(a)+m_1h(t^s)f\left(\frac{b}{m_1}\right),$$

$$g\left(\frac{ab}{ta+(1-t)b}\right)\leq h((1-t)^s)g(a)+m_2h(t^s)g\left(\frac{b}{m_2}\right).$$

Now, using $\langle x_1 \quad x_2, x_3 \quad x_4\rangle \geq 0, (x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2$, $x_3<x_4$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)\left[h((1-t)^s)g(a)+m_2h(t^s)g\left(\frac{b}{m_2}\right)\right]$$

$$+g\left(\frac{ab}{ta+(1-t)b}\right)\left[h((1-t)^s)f(a)+m_1h(t^s)f\left(\frac{b}{m_1}\right)\right]$$

$$\leq\left[h((1-t)^s)f(a)+m_1h(t^s)f\left(\frac{b}{m_1}\right)\right]\left[h((1-t)^s)g(a)+m_2h(t^s)g\left(\frac{b}{m_2}\right)\right]$$

$$+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right).$$

Thus

$$g(a)h((1-t)^s)f\left(\frac{ab}{ta+(1-t)b}\right)+m_2g\left(\frac{b}{m_2}\right)h(t^s)f\left(\frac{ab}{ta+(1-t)b}\right)$$

$$+f(a)h((1-t)^s)g\left(\frac{ab}{ta+(1-t)b}\right)+m_1f\left(\frac{b}{m_1}\right)h(t^s)g\left(\frac{ab}{ta+(1-t)b}\right)$$

$$\leq[h((1-t)^s)]^2[f(a)g(a)]+m_1h(t^s)h((1-t)^s)f\left(\frac{b}{m_1}\right)g(a)$$

$$+m_2h(t^s)h((1-t)^s)f(a)g\left(\frac{b}{m_2}\right)+m_1m_2[h(t^s)]^2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)$$

$$+f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right).$$

Integrating the above inequality with respect to t over $[0,1]$, we have

$$g(a)\int_0^1 h((1-t)^s)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+m_2g\left(\frac{b}{m_2}\right)\int_0^1 h(t^s)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t$$

$$+f(a)\int_0^1 h((1-t)^s)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+m_1f\left(\frac{b}{m_1}\right)\int_0^1 h(t^s)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t$$

$$\leq [f(a)g(a)]\int_0^1 [h((1-t)^s)]^2 \mathrm{d}t + m_1 f\left(\frac{b}{m_1}\right) g(a)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t$$

$$+m_2 f(a) g\left(\frac{b}{m_2}\right)\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t + m_1 m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\int_0^1 [h(t^s)]^2\mathrm{d}t$$

$$+\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\,.$$

This implies

$$\left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(x-\frac{1}{b}\right)^s\right)\left[f(a)g\left(\frac{1}{x}\right)+g(a)f\left(\frac{1}{x}\right)\right]\mathrm{d}x+$$

$$\left(\frac{ab}{b-a}\right)^{s+1}\int_{\frac{1}{b}}^{\frac{1}{a}} h\left(\left(\frac{1}{a}-x\right)^s\right)\left[m_1 f\left(\frac{b}{m_1}\right)g\left(\frac{1}{x}\right)+m_2 g\left(\frac{b}{m_2}\right)f\left(\frac{1}{x}\right)\right]\mathrm{d}x$$

$$\leq\left[f(a)g(a)+m_1m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\int_0^1 [h(t^s)]^2\mathrm{d}t$$

$$+\left[m_1 f\left(\frac{b}{m_1}\right)g(a)+m_2 f(a)g\left(\frac{b}{m_2}\right)\right]\int_0^1 h(t^s)h((1-t)^s)\mathrm{d}t$$

$$+\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}\mathrm{d}x\,,$$

which is the required result.

Lemma 2. Let $f: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m)-convex function, where $s\in(-1,1]$, $m\in(0,1]$, $a,b\in I$ with $a<b$, $ma, \frac{b}{m}\in I$. Then

$$f\left(\frac{abx}{(a+b)x-ab}\right)\leq\left[h(t^s)+h((1-t)^s)\right]\left[f(a)+mf\left(\frac{b}{m}\right)\right]-f(x).$$

Proof. As we know that $x\in[a,b]$ can be represented as

$$x=\frac{ab}{ta+(1-t)b}\,,\ \forall t\in[0,1]\,.$$

Thus

$$f\left(\frac{abx}{(a+b)x-ab}\right)=f\left(\frac{ab}{(1-t)a+tb}\right)$$

$$\leq h(t^s)f(a)+mh((1-t)^s)f\left(\frac{b}{m}\right)$$

$$=\left[h(t^s)+h((1-t)^s)\right]\left[f(a)+mf\left(\frac{b}{m}\right)\right]-\left[h((1-t)^s)f(a)+mh(t^s)f\left(\frac{b}{m}\right)\right]$$

$$\leq\left[h(t^s)+h((1-t)^s)\right]\left[f(a)+mf\left(\frac{b}{m}\right)\right]-f(x)\,.$$

Theorem 6. Let $f: I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic (h,s,m)-convex function, where $s\in(-1,1]$, $m\in(0,1]$, $a,b\in I$ with $a<b$, $ma,\frac{b}{m}\in I$. If $f\in L[ma,\frac{b}{m}]$, then

$$\frac{1}{2h\left(\frac{1}{2^s}\right)} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x \le \frac{1}{2}\int_a^b \frac{\left[f(x)+mf\left(\frac{x}{m}\right)\right]g(x)}{x^2}\mathrm{d}x$$

$$\le \frac{f(a)+mf\left(\frac{b}{m}\right)}{2}\left[h(t^s)+h((1-t)^s)\right]\int_a^b \frac{g(x)}{x^2}\mathrm{d}x$$

$$+\int_a^b \frac{\left[mf\left(\frac{x}{m}\right)-f(x)\right]g(x)}{x^2}\mathrm{d}x,$$

where $g:[a,b]\subset\mathbb{R}\setminus\{0\}$ is nonnegative, integrable and satisfies

$$g(x)=g\left(\frac{abx}{[a+b]x-ab}\right),$$

for all $x\in[a,b]$.

Proof. Using the given fact and Lemma 2, we have

$$\frac{1}{2h\left(\frac{1}{2^s}\right)} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}\mathrm{d}x$$

$$=\frac{1}{2h\left(\frac{1}{2^s}\right)}\int_a^b f\left(\frac{2abx}{(a+b)x-ab+ab}\right)\frac{g(x)}{x^2}\mathrm{d}x$$

$$\le \frac{1}{2h\left(\frac{1}{2^s}\right)}\int_a^b h\left(\frac{1}{2^s}\right)\left[f\left(\frac{abx}{(a+b)x-ab}\right)+mf\left(\frac{x}{m}\right)\right]\frac{g(x)}{x^2}\mathrm{d}x$$

$$=\frac{1}{2}\int_a^b f\left(\frac{abx}{(a+b)x-ab}\right)\frac{g(x)}{x^2}\mathrm{d}x+\frac{m}{2}\int_a^b \frac{f\left(\frac{x}{m}\right)g(x)}{x^2}\mathrm{d}x$$

$$=\frac{1}{2}\int_a^b \frac{\left[f(x)+mf\left(\frac{x}{m}\right)\right]g(x)}{x^2}\mathrm{d}x.$$

To prove the other part of the inequality, we consider

$$\frac{1}{2}\int_a^b \frac{\left[f(x)+mf\left(\frac{x}{m}\right)\right]g(x)}{x^2}\mathrm{d}x$$

$$=\frac{1}{2}\int_a^b f\left(\frac{abx}{(a+b)x-ab}\right)\frac{g(x)}{x^2}\mathrm{d}x+\frac{m}{2}\int_a^b \frac{f\left(\frac{x}{m}\right)g(x)}{x^2}\mathrm{d}x$$

$$\leq \frac{1}{2}\int_a^b \left[h(t^s)+h((1-t)^s)\right]\left[f(a)+mf\left(\frac{b}{m}\right)\right]-f(x)]\frac{g(x)}{x^2}\mathrm{d}x+m\int_a^b \frac{f\left(\frac{x}{m}\right)g(x)}{x^2}\mathrm{d}x$$

$$=\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}\left[h(t^s)+h((1-t)^s)\right]\int_a^b \frac{g(x)}{x^2}\mathrm{d}x+\int_a^b \frac{\left[mf\left(\frac{x}{m}\right)-f(x)\right]g(x)}{x^2}\mathrm{d}x.$$

This completes the proof.

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